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On the dynamics of a finite pre-strained bi-layered slab resting on a rigid foundation under the action of an oscillating moving load

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ABSTRACT

This paper investigates the dynamic response of a finitely pre-strained bi-layered slab resting on a rigid foundation to a time-harmonic oscillating moving load, within the scope of the piecewise-homogeneous body model utilizing the 3D linearized wave propagation theory in the initially stressed body. It is assumed that the materials of the layers are highly elastic ones and their elasticity relations are given in terms of the harmonic potential. Moreover, it is also assumed that the velocity of the line-located time-harmonic oscillating moving load is constant as it acts on the free face of the upper layer of the slab. Our investigations were carried out for a 2D problem (plane-strain state) under subsonic velocity for a moving load in complete contact conditions. The corresponding numerical results were obtained for the stiff upper layer and soft lower layer system in which the constants of elasticity for the upper layer material are greater than those of the lower layer material. Numerical results are presented and discussed for the critical velocity and stress distribution for various values of the problem parameters. In particular, it is established that with the oscillating frequency of the moving load, the values of the critical velocity decrease. Moreover, it is established that the initial stretching of the upper layer of the slab causes the critical velocity to increase and the absolute values of the normal stresses acting on the mid-planes of the layers of the slab to decrease.

1. Introduction

1.1. On the application fields for the model used

A bi-layered slab (with a soft lower and stiff upper layer) resting on a rigid foundation is usually used for modeling floating-slab track systems with continuous slabs which are widely used to control vibration from underground trains [1]. This system is composed of the track which is mounted on a concrete slab resting on rubber bearings, glass fibers or steel springs. Consequently, the principal components relevant to this system are rails, railpads, the floating slab and slab bearings. The track-bed is modeled as a rigid foundation, as the stiffness of the slab bearings is normally much less than the stiffness of the track-bed. At the same time, the system consisting of a bi-layered slab resting on a rigid foundation can be

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considered, in a certain sense, as the generalization of the system consisting of a layer+half-plane which has been used in relevant investigations [2–8]. For example, this system generalization accounts for the waves reflected from the supporting ground.

It is obvious that the total stiffness of the concrete layer containing the track, as well as the total stiffness of the rubber layer containing the glass fibers, is different for different directions, i.e. the total stiffness of the mentioned layers is anisotropic. This anisotropy can be modeled and controlled as initial stresses (strains) in these layers. Moreover, the initial strains can occur by themselves in the layers and in many cases, the magnitude of these stresses has a large effect. The source of these initial stresses or strains could be sharp changes in environmental conditions (i.e. temperature changes) [9] which can cause roadbeds, aircraft runways, etc. to be initially stressed; alternately, the source could be the manufacturing and assembly procedures of the systems considered, the action of the geostatic and geodynamic forces in the Earth's crust (which is modeled as a lower layer or as a half-plane in the corresponding investigations), etc. In fact, it is necessary to take these initial stresses (strains) into account within the study of the dynamic response of vehicle-tracking systems to a moving load. In these cases the excitation of the moving vehicle on the system mentioned is modeled as the action of an oscillating moving load on that system.

Consequently, the model consisting of a finite pre-stained bi-layered slab resting on a rigid foundation under action of an external oscillating moving load has real application possibilities. Nevertheless, the investigations and analyses presented in the present paper will be made without any concretization to the fields of application. However, the obtained results can be used in each aforementioned field under suitable selection of the values of the dimensionless problem parameters.

1.2. A review of published results

We start this review with the investigation made in Ref. [2] in which the dynamical response of the system consisting of a covering layer+half plane to a moving load has been studied. The equations of motion of the covering layer (plate) are described within the scope of the Timoshenko theory, but the equation of motion of the half-plane is described within the scope of the exact linear theory of elastodynamics. The plane-strain state is considered as is the load sinusoidally varying along the load-moving direction; the line load is also examined. It has been assumed that the velocity of the moving load is constant. For the sinusoidally varying load, it was established that for the system considered the resonance type effect occurs in the case where a load of certain wave length moves with a velocity equal to the velocity of free waves of that wave length. For the sinusoidal varying loading the sought values are expressed through $(\Delta(c))^{-1}$, where $\Delta(c)=0$ is a dispersion equation of the previously mentioned waves (here c is a wave velocity). However, under action of the moving line load, the sought values are expressed through $\int_0^{+\infty}(\bullet)(\Delta(x))^{-1}\,\mathrm{d}x$, and therefore the existence of the resonance type effect, expressed through the dispersion equation $\Delta(x)=0$ must have a double real root, i.e. this root must satisfy the equations $\Delta(x)=0$ and $\mathrm{d}\Delta(x)/\mathrm{d}x=0$, simultaneously. As is normal in such cases, the phase velocity has a minimum and this velocity is known as the critical velocity. It is obvious that the mentioned phase velocity equals the group velocity. The numerical examination for the line load has been made for cases involving a relatively soft plate, i.e. for the case where $(G_1/\rho_1)/(G/\rho)<1.0$; $G_1(G)$ and $\rho_1(\rho)$ are the shear modulus and density, respectively, of the covering plate (half-plane) material and a relatively stiff plate, i.e. $(G_1/\rho_1)/(G/\rho)>1.0$. It is established that the critical velocity determined in the foregoing manner arises only for those cases where $(G_1/\rho_1)/(G/\rho)>1.0$, i.e. a relatively stiff plate.

The investigation started in [2], over time, has been improved and developed continuously; the latest iterations of this development, the subjects of which are near to the subject of our present investigation, are described in papers [10,11] and other references listed in these papers. In Ref. [10] the critical velocity of a point-located harmonic varying load moving uniformly and acting on the free face plane of the plate resting on the rigid foundation is investigated. The motion of the plate is described by the 3D equations of the linear theory of elastic waves. The critical velocity is determined as that velocity which is equal to the group velocity. Moreover, in Ref. [10] it is noted that the elastic layer is taken as a model for the track-supporting ballast, but the point-located harmonic varying moving load is taken as a model for train-sleeper excitation. The corresponding boundary value problem is solved by employing exponential Fourier transforms over time and for the spatial coordinates. According to the analyses of the obtained numerical results, it is established that the harmonic variation of the moving load causes two critical velocities to exist; one is lower, but the other is higher, than the Rayleigh wave velocity in the plate material. Note that similar results are also obtained in the present investigation which will be discussed below.

In Ref. [11] the surface ground vibrations for a 2D model are considered, consisting of an elastic layer (possessing a small viscosity), and a beam located inside the layer. It is supposed that the layer and beam are infinitely long in the horizontal direction and rest on the rigid foundation. The motion of the beam is written within the scope of the Euler–Bernoulli model. Moreover, it is assumed that the motion of the structure is caused by a point load, which moves uniformly along the beam; three types of this load are considered, namely a constants load, a harmonically varying load and a stationary random load. Note that all investigations carried out in Ref. [11] are also based on knowledge of the critical velocity which is determined from the intersection of the straight line (known as a "kinematic invariant" [12,13]) and dispersion curves of the corresponding wave propagation. Moreover, under action of the external harmonically varying load the mentioned dispersion curves are plotted both for positive and negative wave numbers, because the harmonically varying load can radiate waves with a negative phase velocity. It should also be noted, in all the investigations discussed above, that the surface displacements are studied as well. But the stress distribution in the constituents of the system considered, and stresses acting on the interface planes, have essentially not been studied. Moreover, in the foregoing studies the reference particularities of these systems are not taken into account, one of which is the initial stresses in the component of those sources which have been discussed above. The first attempt to account for the initial stresses on the values of the critical velocity of the moving load was made in Ref. [9] where a system consisting of an ice plate resting on a water layer was considered. In this case the motion of the plate is described within the scope of the Kirchhoff theory and it is established that the initial stretching (compression) causes an increase (decrease) in the values of the critical velocity.

It is obvious that more accurate and trustworthy results for the types of problems related to initially stressed systems can be attained within the scope of the 3D linearized theory of elastic waves in initially stressed bodies (TLTEWISB). The construction of TLTEWISB field equations and their application to wave propagation problems are detailed in [14–21]. Furthermore, through the application of TLTEWISB, the time-harmonic stress field in layered pre-strained bodies has been studied in Refs. [22–25]. A more detailed review of these investigations is given in Refs. [26,27].

Nevertheless, within the framework of TLTEWISB, few studies have been done on the dynamic response to the moving load of a pre-strained layered half-space (Refs. [3–6]). In Ref. [3], as in Ref. [2], the dynamic response was considered for a system consisting of a covering layer and pre-strained half-plane. The equation of motion for the covering layer was described by Timoshenko plate theory, but the equation of motion for the half plane was described by TLTEWISB. The solution to the corresponding boundary value problem was determined by using the exponential Fourier integral transformation. Corresponding numerical investigations were made for the case where constitutive relations for the half-plane material were described in terms of harmonic potential. Moreover, it was assumed that the speed of the moving load was constant and the subsonic case had been taken into consideration. These numerical investigations led to further study of the parameters' influence on the critical velocity in the second study (carried out in Ref. [4]) utilizing the complex potentials of TLTEWISB. As in [2], the numerical results attained in [3] show that a critical velocity occurs for a relatively stiff plate.

In Refs. [5,6], the investigations in [3,4] were developed for the case where the velocity of the moving load is supersonic. This development was made for an incompressible (Ref. [5]) as well as for compressible materials (Ref. [6]). All investigations carried out in Refs. [3–6] were made for 2D problems (plane-strain state).

The investigation carried out in Ref. [7] employed the findings of Refs. [3,4] in developing a case where the covering layer is also strained initially, and where the equation of motion for this layer is also described by TLTEWISB; from this, the influence of the problem parameters on the critical velocity was studied. However, in [7] it is assumed that the materials of the covering and half-plane are isotropic. This assumption significantly restricts theoretical investigations in terms of controlling the critical velocity values for the moving load and the stresses acting on the interface plane through the mechanical properties of the layer and half-plane materials. Therefore, the study made in Ref. [8] further develops the investigation detailed in [7] for the case where the materials of the covering layer and half-plane are anisotropic (orthotropic). The results of the investigations carried out in Refs. [7,8] also show that a critical velocity occurs in the case where the modulus of elasticity in the load moving direction is greater than that of the half-plane material.

1.3. The scope of the present paper

As noted in Refs. [1,10,11,28] and many others, in reality high-speed trains, cars and other high speed transportation vehicles modeled as moving loads are accompanied by their own oscillations. To determine how these accompanying oscillations act on the dynamic response of the pre-strained system considered requires corresponding additional investigations. These investigations are the scope of the present paper. A finite pre-strained bi-layered slab resting on a rigid foundation is considered herein because, as has been noted already, the system consisting of a bi-layered slab and rigid foundation, in many cases, is a more general and appropriate one for the modeling of roads than the system consisting of a covering layer and half-space. The investigations are carried out for 2D problems, i.e. for the plane strain state.

The other new aspect of the present investigation is the following. In Refs. [7,8] it was assumed that the initial strains in the components of the system considered are small strains and the initial stress–strain state is determined within the linear theory of elasticity. However, in the present paper it is assumed that the initial strains in the mentioned components are finite ones and the initial stress–strain state is determined within the scope of the nonlinear theory of elasticity. In this case it is assumed that the materials of the layers of the slab are highly elastic and these layers are finitely pre-strained. At the same time, it is assumed that the mechanical relations of these materials are defined by the harmonic potential. Thus, the main differences between the present work and the previous investigations [3–8] are related to the moving load dynamics acting on a pre-stressed layered medium and are carried out within the scope of the TLTEWISB. These differences can be summarized as follows:

(i) in the present work the dynamics of an oscillating moving load are studied, but in the previous ones the dynamics of a constant moving load were considered;
(ii) in the present work the subject of our investigations is a bi-layered slab resting on a rigid foundation, but in the previous works the subject of the investigations was a system consisting of a covering layer and half-plane;

(iii) in the present work it is assumed that the initial strains exist in each constituent of the system considered and these strains are finite ones, but in the previous investigations (as in [7,8]) it was assumed that the initial strains were small ones, i.e. determined within the scope of the linear theory of elasticity, or (as in [3–6]) it was assumed that the initial strains were finite ones but that these strains existed in the half-plane only.

Throughout the paper, repeated indices indicate a summation over their ranges. However, underlined repeated indices are not to be taken as sums.

2. Formulation of the problem

A bi-layered slab resting on a rigid foundation is considered herein (Fig. 1). Assume that in the natural state the thickness of the upper and lower layers of the slab are $H^{(2)}$ and $H^{(1)}$, respectively. In the natural state, we determine the positions of the points of the layers by the Lagrangian coordinates in the Cartesian system of coordinates $Ox_1x_2x_3$. Suppose that the layers of the slab have infinite length in the directions of the Ox_1 and Ox_3 axes. The Ox_3 axis extends along a direction perpendicular to the plane Ox_1x_2 in Fig. 1 and therefore is not shown in this figure.

We propose that the layers, before being compounded with each other and with a rigid foundation, be stretched separately along the Ox_1 axis direction and that in each of them, the homogeneous initial finite strain state appear. These initial strains are caused by the static forces acting in the Ox_1 axis direction at infinity. Note that the action of the forces continues all further dynamic processes.

With the initial state of the layers of the slab we associate the Lagrangian Cartesian system of coordinates $Oy_1y_2y_3$ and it is supposed that the origin of this system coincides with the origin of the system $Ox_1x_2x_3$, and the coordinate axes Oy_1, Oy_2 and Oy_3 coincide with the coordinate axes Ox_1, Ox_2 and Ox_3, respectively. Assuming that the material of the layers is compressible, the elasticity relations are given through the harmonic potential. Moreover, it is assumed that the values related to the upper and lower layers of the slab are denoted by upper indices (2) and (1), respectively. Furthermore, the values related to the initial state are denoted by the upper index 0. To simplify further discussions, a local system of coordinates $O^{(m)}x_1^{(m)}x_2^{(m)}x_3^{(m)}$ (in the natural state) and $O^{(m)}y_1^{(m)}y_2^{(m)}y_3^{(m)}$ (in the initial state) has been associated with the middle plane of the m-th layer. The system of coordinates $O^{(m)}x_1^{(m)}x_2^{(m)}x_3^{(m)}$ ($O^{(m)}y_1^{(m)}y_2^{(m)}y_3^{(m)}$) is attained from the system of coordinates $Ox_1x_2x_3(Oy_1y_2y_3)$ by parallel transfer along the Ox_2 (Oy_2) axis.

Thus, according to the foregoing, the initial state in the layers can be determined as follows:

$$u_{\underline{i}}^{(m),0} = (\lambda_{\underline{i}}^{(m)} - 1)x_{\underline{i}}^{(m)}, \quad \lambda_{\underline{i}}^{(m)} = const_{im}, \quad y_{\underline{i}}^{(m)} = \lambda_{\underline{i}}^{(m)} x_{\underline{i}}^{(m)}, \tag{1}$$

where $u_i^{(m),0}$ is a component of the displacement vector in the m-th layer in the initial strain state and $\lambda_i^{(m)}$is an elongation factor which characterizes the change in the length of the line element in the Ox_i axis direction. This parameter is determined by the expression $\lambda_i^{(m)} = \sqrt{1 + 2\varepsilon_i^{(m)}}$, where $\varepsilon_i^{(m)}$is an i-th principal value of the Green's strain tensor of the m-th layer. The expression of the components of this tensor through the components of the displacement vector will be given below.

Within this, let us investigate the mechanical behavior of the considered slab in the case where on the free face plane of the upper layer, the line-located normal time-harmonic moving force acts. This investigation will be made by the use of coordinates associated with the initial state, i.e. by the use of coordinates $y_i^{(m)}$, in the framework of the TLTEWISB.

In the construction of the field equations of the TLTEWISB, one considers two states of a deformable solid. The first is regarded as the initial or unperturbed state and the second is a perturbed state with respect to the unperturbed one. By the "state of a deformable solid" both motion and equilibrium (as a particular case of motion) is meant. It is assumed that all values in a perturbed state can be represented as a sum of the values in the initial state and the perturbations. The latter is also assumed to be small in comparison with the corresponding values in the initial state. It is also assumed that both initial (unperturbed) and perturbed states are described by the equations of nonlinear solid mechanics. Owing to the fact that the perturbations are small, the relationships for the perturbed state in the vicinity of appropriate values for the unperturbed state are linearized, and then the relations for the perturbed state are subtracted from them. The result is the

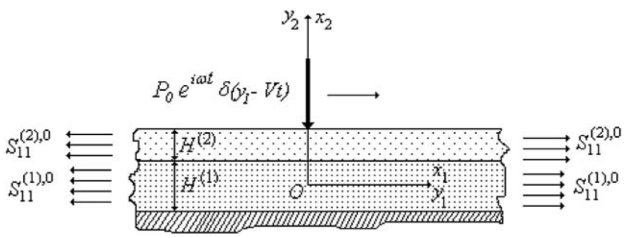

Fig. 1. The geometry of a bi-layered slab resting on a rigid foundation.

equations of the TLTEWISB. The general problems of the TLTEWISB have been elaborated in many investigations such as Refs. [14–19] and others. In the present paper, we will follow the style and notation used in monograph [19].

Thus, the following are the basic relations of the TLTEWISB for the compressible body under the plane-strain state in the Oy_1y_2 plane. These relations are satisfied within each layer because we use the piecewise homogeneous body model.

The equation of motion is

$$\frac{\partial Q_{ij}^{(m)}}{\partial y_j^{(m)}} = \rho^{(m)} \frac{\partial^2 u_i'^{(m)}}{\partial t^2}, \tag{2}$$

and the mechanical relations are

$$Q_{ij}^{(m)} = \omega_{ij\alpha\beta}^{(m)} \frac{\partial u_\alpha'^{(m)}}{\partial y_\beta^{(m)}}, \tag{3}$$

where

$$\omega_{1111}^{(m)} = \frac{\lambda_1^{(m)}}{\lambda_2^{(m)}}(\lambda^{(m)} + 2\mu^{(m)}), \quad \omega_{2222}^{(m)} = \frac{\lambda_2^{(m)}}{\lambda_1^{(m)}}(\lambda^{(m)} + 2\mu^{(m)}), \quad m = 1, 2,$$

$$\omega_{1122}^{(m)} = \omega_{2211}^{(m)} = \lambda^{(m)}, \quad \omega_{1212}^{(m)} = \omega_{2121}^{(m)} = \frac{2\lambda_2^{(m)}\mu^{(m)}}{\lambda_1^{(m)} + \lambda_2^{(m)}}, \quad \omega_{1221}^{(m)} = \omega_{2112}^{(m)} = \frac{2(\lambda_2^{(m)})^2\mu^{(m)}}{\lambda_2^{(m)}(\lambda_1^{(m)} + \lambda_2^{(m)})}. \tag{4}$$

In Eqs. (3) and (4) the following notation is used: $Q_{ij}^{(m)}$ are the perturbations of the components of the Kirchhoff stress tensor in the m-th layer related to the areas of the initial state, $u_j'^{(m)}$ are the components of the perturbations of the displacement vector, and $\rho^{(m)}$ and $\lambda^{(m)}$, $\mu^{(m)}$ are the densities also related to the volume of the initial state and mechanical constants of the m-th material.

Note that the constants $\omega_{ij\alpha\beta}^{(m)}$ in (3) and (4) are determined through the initial strain state (1) and through the elastic potential. As has been noted above, in the present work the elasticity relations of the layers' materials are determined by harmonic potential. This potential is given as follows:

$$\Phi^{(m)} = \tfrac{1}{2}\lambda^{(m)}(s_1^{(m)})^2 + \mu^{(m)}s_2^{(m)}, \tag{5}$$

where

$$s_1^{(m)} = \left(\sqrt{1 + 2\varepsilon_1^{(m)}} - 1\right) + \left(\sqrt{1 + 2\varepsilon_2^{(m)}} - 1\right) + \left(\sqrt{1 + 2\varepsilon_3^{(m)}} - 1\right),$$

$$s_2^{(m)} = \left(\sqrt{1 + 2\varepsilon_1^{(m)}} - 1\right)^2 + \left(\sqrt{1 + 2\varepsilon_2^{(m)}} - 1\right)^2 + \left(\sqrt{1 + 2\varepsilon_3^{(m)}} - 1\right)^2. \tag{6}$$

In (6) $\varepsilon_i^{(m)}(i = 1, 2, 3)$ are the principal values of the Green's strain tensor.

Let us consider briefly the definition of the stress and strain tensors in the large elastic deformation theory which are used in the present investigation. For this purpose we use the Lagrange coordinates x_i $(i = 1, 2, 3)$ in the Cartesian system of coordinates $Ox_1x_2x_3$ and the position of the points after and before deformations we determine by the vectors $\mathbf{r}^*$ and $\mathbf{r}$ where $\mathbf{r}^* = \mathbf{r} + \mathbf{u}$. Here $\mathbf{u} = u_i\mathbf{g}_i$ is a displacement vector expressed by the unit basic vectors $\mathbf{g}_i$. Taking the relations $\mathrm{d}\mathbf{r}^* \cdot \mathrm{d}\mathbf{r}^* = \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} + 2\,\mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{u} + \mathrm{d}\mathbf{u} \cdot \mathrm{d}\mathbf{u}$ (here the symbol " $\cdot$ " means the scalar product of the vectors), $\mathrm{d}\mathbf{u} \cdot \mathrm{d}\mathbf{u} = (\partial u_k/\partial x_i)(\partial u_k/\partial x_j)\,\mathrm{d}x_i\,\mathrm{d}x_j$ and $2\,\mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{u} = 2(\partial u_k/\partial x_i)\,\mathrm{d}x_k\,\mathrm{d}x_i$ into account, it can be written that $\mathrm{d}\mathbf{r}^* \cdot \mathrm{d}\mathbf{r}^* - \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} = 2\varepsilon_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j$, where

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_n}{\partial x_j}\frac{\partial u_n}{\partial x_j}\right). \tag{7}$$

This is a component of the Green's strain tensor $\tilde{\boldsymbol{\varepsilon}}$ which is symmetric.

Let us consider the definition of the Kirchhoff stress tensor. The use of various types of stress tensors in the large (finite) elastic deformation theory is connected with the reference of the components of these tensors to the unit area of the relevant surface elements in the deformed or un-deformed state, because, in contrast to the linear theory of elasticity, in the finite elastic deformation theory the difference between the areas of the surface elements taken before and after deformation must be accounted for in the derivation of the equation of motion and under satisfaction of the boundary conditions. According to the aim of the present investigation, we here consider two types of stress tensors denoted by $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{S}}$ the components of which refer to the unit area of the relevant surface elements in the un-deformed state, but act on the surface elements in the deformed state. The components S_{ij} of the stress tensor $\tilde{\mathbf{S}}$ are determined through the strain energy potential $\Phi = \Phi(\varepsilon_{11}, \varepsilon_{22}, \ldots, \varepsilon_{33})$, where ε_{ij} is a component of the Green's strain tensor (7), by the use of the following expression:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial \varepsilon_{ij}} + \frac{\partial}{\partial \varepsilon_{ji}}\right)\Phi(\varepsilon_{11}, \varepsilon_{22}, \ldots, \varepsilon_{33}). \tag{8}$$

The components q_{ij} of the stress tensor $\tilde{\mathbf{q}}$ are determined by the expression

$$q_{ij} = \left(\delta_k^j + \frac{\partial u_j}{\partial x_k}\right)S_{ik}. \tag{9}$$

Here δ_k^j is the Kronecker symbol. The stress tensor $\tilde{\mathbf{q}}$ with components determined by expressions (8) and (9) is called the Kirchhoff stress tensor. According to expressions (7)–(9), the stress tensor $\tilde{\mathbf{S}}$ is symmetric, but the Kirchhoff stress tensor $\tilde{\mathbf{q}}$ is non-symmetric. Thus, with this we restrict ourselves to consideration of the definition of the stress and strain tensors in the finite elastic deformation theory. These definitions are given without any restriction related to the association of the selected coordinate systems to the natural or initial state. However, in using the coordinate system associated with the initial deformed state, the initial strain state can be taken as an "un-deformed" state in the foregoing definitions.

Now we attempt to attain the Eqs. (3) and (4) by employing a linearization procedure. Throughout this procedure in order to simplify the writing of the mathematical symbols, as in the foregoing Eqs. (7), (8) and (9) we will omit the upper index (m). Nevertheless the results will be simultaneously used for each component of the system considered by supplying them with the upper index (1) or (2). But in order to denote the displacements, strains and stresses regarding the initial strain state as above we will use the upper index "0".

Thus, according to (1), (5)–(9), we attain that

$$S_{\underline{i}\underline{i}}^0 = [\lambda(\lambda_1 + \lambda_2 - 2) + 2\mu(\lambda_{\underline{i}} - 1)](\lambda_{\underline{i}})^{-1}, \quad S_{12}^0 = 0. \tag{10}$$

It follows from the statement of the problem that $S_{22}^0 = 0$, according to which, the following expression for λ_2 is attained:

$$\lambda_2 = [2\mu - \lambda(\lambda_1 - 2)](\lambda + 2\mu)^{-1}. \tag{11}$$

In this way, for selected layer materials, the magnitude of the initial strains and the initial stresses in them can be determined only through the parameter λ_1. In this case the perturbation of the components of the non-symmetric Kirchhoff stress tensor q_{ij} (denoted by q'_{ij}) related to the areas of the natural state are determined by the following expression:

$$q'_{ij} = \left(\delta_k^j + \frac{\partial u_i 0}{\partial x_k}\right)S'_{ik} + S_{ik}^0\frac{\partial u'_j}{\partial x_k}, \tag{12}$$

where S'_{ik} is a perturbation of the components of the foregoing symmetric stress tensor $\tilde{\mathbf{S}}$.

By the linearization of Eq. (8), the following relation is obtained for these components:

$$S'_{in} = \left\{\frac{1}{4}\left(\delta_k^\beta + \frac{\partial u_\beta^0}{\partial x_k}\right)\left(\frac{\partial}{\partial \varepsilon_{k\beta}^0} + \frac{\partial}{\partial \varepsilon_{\beta k}^0}\right)\left(\frac{\partial}{\partial \varepsilon_{in}^0} + \frac{\partial}{\partial \varepsilon_{ni}^0}\right)\Phi^0\right\}\frac{\partial u'_\alpha}{\partial x_\beta}. \tag{13}$$

Using the relations

$$\begin{aligned} &Q_{11}\,\mathrm{d}y_2\,\mathrm{d}y_3 = q'_{11}\,\mathrm{d}x_2\,\mathrm{d}x_2, \ldots, Q_{21}\,\mathrm{d}y_1\,\mathrm{d}y_3 = q'_{21}\,\mathrm{d}x_1\,\mathrm{d}x_3, \quad \mathrm{d}y_{\underline{i}} = \lambda_{\underline{i}}\,\mathrm{d}x_{\underline{i}}, \quad \lambda_3 = 1.0 \\ &\Rightarrow Q_{11} = q'_{11}/\lambda_2, \ldots, Q_{21} = q'_{21}/\lambda_1. \end{aligned} \tag{14}$$

and changing $\partial u'_j/\partial x_k$ and $\partial u'_\alpha/\partial x_\beta$ in Eqs. (12) and (13) with $\lambda_k \partial u'_j/\partial y_k$ and $\lambda_\beta \partial u'_\alpha/\partial y_\beta$, respectively, we attain Eqs. (3) and (4) from Eqs. (12) and (13) after some mathematical manipulations. Next we consider the obtainment of the expressions for Q_{11}, ω_{1111} and ω_{1122} given in Eqs. (3) and (4). From Eqs. (1), (12) and (13) it can be easily attained that

$$q'_{11} = \lambda_1 S'_{11} + S_{11}^0\frac{\partial u'_1}{\partial x_1}, \quad S'_{11} = \lambda_1\frac{\partial}{\partial \varepsilon_{11}^0}S_{11}^0\frac{\partial u'_1}{\partial x_1} + \lambda_2\frac{\partial}{\partial \varepsilon_{22}^0}S_{11}^0\frac{\partial u'_2}{\partial x_2}. \tag{15}$$

Taking the relations

$$\lambda_1\frac{\partial}{\partial \varepsilon_{11}^0}S_{11}^0 = \frac{\lambda_1}{\lambda_1}\frac{\partial S_{11}^0}{\partial \lambda_1} = \frac{1}{\lambda_1}(\lambda + 2\mu) - \frac{1}{(\lambda_1)^2}S_{11}^0, \quad \lambda_2\frac{\partial}{\partial \varepsilon_{22}^0}S_{11}^0 = \frac{\lambda_2}{\lambda_2}\frac{\partial S_{11}^0}{\partial \lambda_2} = \frac{1}{\lambda_1}\lambda, \tag{16}$$

which are obtained from the definition of the parameter λ_i and the expression for S_{11}^0 in Eq. (10), and the relations (15) into account, the following mathematical transformations can be made:

$$q'_{11} = (\lambda + 2\mu)\frac{\partial u'_1}{\partial x_1} + \lambda\frac{\partial u'_2}{\partial x_2} = \lambda_1(\lambda + 2\mu)\frac{\partial u'_1}{\partial y_1} + \lambda_2\lambda\frac{\partial u'_2}{\partial y_2},$$

$$Q_{11} = q'_{11}/\lambda_2 = \frac{\lambda_1}{\lambda_2}(\lambda+2\mu)\frac{\partial u'_1}{\partial y_1} + \lambda\frac{\partial u'_2}{\partial y_2} = \omega_{1111}\frac{\partial u'_1}{\partial y_1} + \omega_{1122}\frac{\partial u'_2}{\partial y_2},$$
$$\Rightarrow \omega_{1111} = \frac{\lambda_1}{\lambda_2}(\lambda+2\mu), \quad \omega_{1122} = \lambda. \tag{17}$$

By supplying the foregoing expressions for ω_{1111} and ω_{1122} in (17) with the upper index (m) we attain the expressions for $\omega^{(m)}_{1111}$ and, $\omega^{(m)}_{1122}$, respectively, given in Eq. (4). In a similar manner we can obtain the expressions of the remaining components $\omega^{(m)}_{ij\alpha\beta}$ which enter Eqs. (3) and (4). Thus with this we restrict ourselves to consideration of the basic equations and relations within the scope of which the present investigation is carried out.

Consider the contact and boundary conditions. The considered system is excited by a line-located time-harmonic oscillating moving load on the upper layer; therefore the following conditions must be satisfied:

$$Q^{(2)}_{21}|_{y^{(2)}_2=\lambda^{(2)}_2 H^{(2)}/2} = 0, \quad Q^{(2)}_{22}|_{y^{(2)}_2=\lambda^{(2)}_2 H^{(2)}/2} = -P_0 e^{i\omega t}\delta(y_1 - Vt), \tag{18}$$

$$\{u'^{(2)}_i; Q^{(2)}_{2i}\}|_{y^{(2)}_2=-\lambda^{(2)}_2 H^{(2)}/2} = \{u'^{(1)}_i; Q^{(1)}_{2i}\}|_{y^{(1)}_2=+\lambda^{(1)}_2 H^{(1)}/2}, \quad u'^{(1)}_i|_{y^{(1)}_2=-\lambda^{(1)}_2 H^{(1)}/2} = 0. \tag{19}$$

In (18) V and ω denote the velocity and frequency of the moving load with amplitude P_0.

This completes the formulation of the problem. It should be noted that in the case where $\lambda^{(m)}_i = 1.0$ $(i=1,2;\ m=1,2)$ the formulation described above transforms into the corresponding one within the scope of the classical linear theory of elastodynamics.

3. Method of solution

Below we will only deal with perturbation $u'^{(m)}_i$ and $Q^{(m)}_{ik}$, and will omit the upper prime in notation $u'^{(m)}_i$, i.e. instead of the notation $u'^{(m)}_i$ we will use the notation $u^{(m)}_i$.

By using the coordinate system

$$y'_1 = y'^{(1)}_1 = y'^{(2)}_1 = y_1 - Vt, \quad y'^{(m)}_2 = y^{(m)}_2, \tag{20}$$

which moves with the loading force and represents the sought values as

$$g(y'^{(m)}_1, y'^{(m)}_2, t) = \bar{g}(y'^{(m)}_1, y'^{(m)}_2)e^{i\omega t} \tag{21}$$

from Eqs. (2)–(4), the following equations of motion in terms of displacement are obtained:

$$\omega'^{(\underline{m})}_{kj\alpha\beta}\frac{\partial^2 u^{(\underline{m})}_\alpha}{\partial y^{(\underline{m})}_k \partial y^{(\underline{m})}_\beta} = \frac{1}{(c^{(\underline{m})}_2)^2}\left(V^2\frac{\partial^2 u^{(\underline{m})}_j}{\partial (y^{(m)}_1)^2} - 2i\omega V\frac{\partial u^{(\underline{m})}_j}{\partial y^{(\underline{m})}_1} - \omega^2\frac{\partial^2 u^{(\underline{m})}_j}{\partial (y^{(\underline{m})}_2)^2}\right), \quad j=1,2, \quad i=\sqrt{-1}, \tag{22}$$

where

$$\omega'^{(\underline{m})}_{kj\alpha\beta} = \frac{\omega^{(\underline{m})}_{kj\alpha\beta}}{\mu^{(\underline{m})}}, \quad c^{(\underline{m})}_2 = \sqrt{\frac{\mu^{(\underline{m})}}{\rho^{(\underline{m})}}}. \tag{23}$$

In Eq. (22), the upper prime in y_1 and y_2, and over bar in $u^{(m)}_1$ and $u^{(m)}_2$are omitted. In this case, the second boundary condition (18) is replaced by the following:

$$Q^{(2)}_{22}\Big|_{y^{(2)}_2=\lambda^{(2)}_2 H^{(2)}/2} = -P_0\delta(y_1). \tag{24}$$

Thus, the other conditions in (18) and (19) are also valid for the new coordinate system (20) and for the amplitude of the sought values.

Consider the solutions to Eq. (22). For this purpose we employ the exponential Fourier transformation with respect to the $y_1$coordinate defined as

$$f_F(s, y_2) = \int_{-\infty}^{+\infty} f(y_1, y_2)e^{-isy_1}\,dy_1 \tag{25}$$

in Eq. (22) and given the corresponding boundary and contact conditions. As a result of this transformation the following equations with respect to $u^{(\underline{m})}_{1F}(s, y^{(\underline{m})}_2)$ and $u^{(\underline{m})}_{2F}(s, y^{(\underline{m})}_2)$ are attained from (22):

$$(-\psi^{(\underline{m})} - \bar{s}^2\omega'^{(\underline{m})}_{1111})u^{(\underline{m})}_{1F} + \omega'^{(\underline{m})}_{2112}\frac{d^2 u^{(\underline{m})}_{1F}}{d(\bar{y}^{(\underline{m})}_2)^2} + (\omega'^{(\underline{m})}_{1122} + \omega'^{(\underline{m})}_{2121})i\bar{s}\frac{du^{(\underline{m})}_{2F}}{d\bar{y}^{(\underline{m})}_2} = 0,$$

$$(-\psi^{(m)} - \bar{s}^2\omega'^{(m)}_{1221})u^{(m)}_{2F} + \omega'^{(m)}_{2222}\frac{d^2u^{(m)}_{2F}}{d(\bar{y}^{(m)}_2)^2} + (\omega'^{(m)}_{1212} + \omega'^{(m)}_{2211})i\bar{s}\frac{du^{(m)}_{1F}}{d\bar{y}^{(m)}_2} = 0, \quad (26)$$

where

$$\psi^{(m)} = \frac{(c_2^{(2)})^2}{(c_2^{(m)})^2}(-\bar{s}^2c^2 + 2\Omega c\bar{s} - \Omega^2), \quad c = \frac{V}{c_2^{(2)}}, \quad \Omega = \frac{\omega H^{(2)}}{c_2^{(2)}}, \quad \bar{s} = sH^{(2)}. \quad (27)$$

From the second equation in (26), it can be written that

$$\frac{du^{(m)}_{1f}}{d\bar{y}^{(m)}_2} = i\left(a^{(m)}u^{(m)}_{2F} + b^{(m)}\frac{d^2u^{(m)}_{2F}}{d(\bar{y}^{(m)}_2)^2}\right), \quad \frac{d^3u^{(m)}_{1f}}{d(\bar{y}^{(m)}_2)^3} = i\left(a^{(m)}\frac{d^2u^{(m)}_{2F}}{d(\bar{y}^{(m)}_2)^2} + b^{(m)}\frac{d^4u^{(m)}_{2F}}{d(\bar{y}^{(m)}_2)^4}\right). \quad (28)$$

Similarly it can also be written from the first equation in (26) that

$$(-\psi^{(m)} - \bar{s}^2\omega'^{(m)}_{1111})\frac{du^{(m)}_{1F}}{d\bar{y}^{(m)}_2} + \omega'^{(m)}_{2112}\frac{d^3u^{(m)}_{1F}}{d(\bar{y}^{(m)}_2)^3} + (\omega'^{(m)}_{1122} + \omega'^{(m)}_{2121})i\bar{s}\frac{d^4u^{(m)}_{2F}}{d(\bar{y}^{(m)}_2)^4} = 0. \quad (29)$$

Substituting the expressions in Eq. (28) into Eq. (29), the equation

$$\frac{d^4u^{(m)}_{2F}}{d(\bar{y}^{(m)}_2)^4} + a_1^{(m)}\frac{d^2u^{(m)}_{2F}}{d(\bar{y}^{(m)}_2)^2} + b_1^{(m)}u^{(m)}_{2F} = 0, \quad (30)$$

is obtained, where

$$a^{(m)} = \frac{-(\psi^{(m)} + \bar{s}^2\omega'^{(m)}_{1221})}{\bar{s}(\omega'^{(m)}_{1212} + \omega'^{(m)}_{2211})}, \quad b^{(m)} = \frac{\omega'^{(m)}_{2222}}{\bar{s}(\omega'^{(m)}_{1212} + \omega'^{(m)}_{2211})},$$

$$a_1^{(m)} = [b^{(m)}(-\psi^{(m)} - \bar{s}^2\omega'^{(m)}_{1111}) + a^{(m)}\omega'^{(m)}_{2112} + \bar{s}(\omega'^{(m)}_{1122} + \omega'^{(m)}_{2121})](\omega'^{(m)}_{2112}b^{(m)})^{-1},$$

$$b_1^{(m)} = a^{(m)}(-\psi^{(m)} - \bar{s}\omega'^{(m)}_{1111})(\omega'^{(m)}_{2112}b^{(m)})^{-1}. \quad (31)$$

Thus, we find the solution to Eq. (30) as follows:

$$u^{(m)}_{2F} = A_1^{(m)}(s)e^{K_1^{(m)}\bar{y}_2^{(m)}} + A_2^{(m)}(s)e^{-K_1^{(m)}\bar{y}_2^{(m)}} + A_3^{(m)}(s)e^{K_2^{(m)}\bar{y}_2^{(m)}} + A_4^{(m)}(s)e^{-K_2^{(m)}\bar{y}_2^{(m)}}, \quad (32)$$

where

$$K_1^{(m)} = \sqrt{-\frac{a_1^{(m)}}{2} + d_1^{(m)}}, \quad K_2^{(m)} = \sqrt{-\frac{a_1^{(m)}}{2} - d_1^{(m)}}, \quad d_1^{(m)} = \sqrt{\frac{(a_1^{(m)})^2}{4} - b_1^{(m)}}. \quad (33)$$

As the subsonic case is considered, therefore it is assumed that the conditions

$$\min\left\{\frac{V}{c_2^{(1)}}, \frac{V}{c_2^{(2)}}\right\} < \min\{\omega'^{(1)}_{ij\alpha\beta}, \omega'^{(2)}_{ij\alpha\beta}\} \quad (34)$$

satisfy each combination of the indices $ij\alpha\beta$. According to (34), it follows from (31) and (32) that in the case where $\Omega = 0$ the values of $K_1^{(m)}$ and $K_2^{(m)}$ in (33) are real values and $K_1^{(m)} > 0$, $K_2^{(m)} > 0$. But in the case where $\Omega > 0$, the values of $K_1^{(m)}$ and $K_2^{(m)}$ can also be complex (most probably pure imaginary) numbers. Note that these statements have been taken into account in developing the calculation algorithm and in constructing the corresponding PC programs.

Thus, from (32), (28), (3) and (4) we have completely determined the Fourier transformation of all values. The corresponding closed system of algebraic equations is obtained from boundary conditions (18), (24) and contact conditions (19) for determination of the unknowns $A_1^{(1)}(s), \ldots, A_4^{(1)}(s)$, $A_1^{(2)}(s), \ldots, A_4^{(2)}(s)$ which enter into these transformations. From the algebraic equations we find the aforementioned unknowns and, by employing the inverse transform

$$f(y_1, y_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_F(s, y_2)e^{isy_1}\,ds. \quad (35)$$

We determine the sought stresses and displacements.

4. General remarks on the Doppler effect and on the algorithm used for obtaining numerical results

In the case where $\Omega = 0$ (as in Refs. [7,8]) or in the case where $c = 0$ (as in Refs. [22,25]) the integral (35) can be reduced to the calculation of either of the integrals $1/\pi \int_{-\infty}^{+\infty} f_F(s,y_2)\cos(sy_1)\,\mathrm{d}s$ (for $u_2^{(m)}, Q_{22}^{(m)}, Q_{11}^{(m)}, \varepsilon_{22}^{(m)}, \varepsilon_{11}^{(m)}$) or $1/\pi \int_{-\infty}^{+\infty} f_F(s,y_2)\sin(sy_1)\,\mathrm{d}s$ (for $u_1^{(m)}, Q_{21}^{(m)}, Q_{12}^{(m)}, \varepsilon_{12}^{(m)}$). However, in the case where $\Omega \times c \neq 0$, this reduction is violated by the term $2\Omega c\bar{s}$ which enters the expression of $\psi^{(m)}$ in (27). Consequently, given the calculation of integral (35) we must use the relation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}(\bullet)\mathrm{e}^{\mathrm{i}sy_1}\,\mathrm{d}s \approx \frac{1}{2\pi}\int_{-S^*}^{+S^*}(\bullet)\cos(sy_1)\,\mathrm{d}s + \frac{\mathrm{i}}{2\pi}\int_{-S^*}^{+S^*}(\bullet)\sin(sy_1)\,\mathrm{d}s. \tag{36}$$

The values of S^* in (36) are determined from the corresponding numerical convergence criterion. At the same time, we use the following notation:

$$Q_{ijc}^{(m)} \approx \frac{1}{2\pi}\int_{-S^*}^{+S^*} Q_{ijF}^{(m)}\cos(sy_1)\,\mathrm{d}s, \quad Q_{ijs}^{(m)} \approx \frac{1}{2\pi}\int_{-S^*}^{+S^*} Q_{ijF}^{(m)}\sin(sy_1)\,\mathrm{d}s,$$

$$u_{ic}^{(m)} \approx \frac{1}{2\pi}\int_{-S^*}^{+S^*} u_{iF}^{(m)}\cos(sy_1)\,\mathrm{d}s, \quad u_{is}^{(m)} \approx \frac{1}{2\pi}\int_{-S^*}^{+S^*} u_{iF}^{(m)}\sin(sy_1)\,\mathrm{d}s,$$

$$\{\tilde{Q}_{22}^{(m)};\tilde{Q}_{11}^{(m)};\tilde{u}_2^{(m)}\} = \{|\tilde{Q}_{22}^{(m)}|\mathrm{e}^{\mathrm{i}\alpha_{22}^{(m)}};|\tilde{Q}_{11}^{(m)}|\mathrm{e}^{\mathrm{i}\alpha_{11}^{(m)}};|\tilde{u}_2^{(m)}|\mathrm{e}^{\mathrm{i}\alpha_2^{(m)}}\},$$

$$\{\tilde{Q}_{21}^{(m)};\tilde{Q}_{12}^{(m)};\tilde{u}_1^{(m)}\} = \{\mathrm{i}|\tilde{Q}_{21}^{(m)}|\mathrm{e}^{\mathrm{i}\alpha_{21}^{(m)}};\mathrm{i}|\tilde{Q}_{12}^{(m)}|\mathrm{e}^{\mathrm{i}\alpha_{12}^{(m)}};\mathrm{i}|\tilde{u}_1^{(m)}|\mathrm{e}^{\mathrm{i}\alpha_1^{(m)}}\}, \tag{37}$$

where

$$|\tilde{Q}_{ij}^{(m)}| = \sqrt{(Q_{ijc}^{(m)})^2 + (Q_{ijs}^{(m)})^2}, \quad |\tilde{u}_i^{(m)}| = \sqrt{(u_{ic}^{(m)})^2 + (u_{is}^{(m)})^2},$$

$$\tan\alpha_{ij}^{(m)} = \frac{Q_{ijs}^{(m)}}{Q_{ijc}^{(m)}}, \quad \tan\alpha_i^{(m)} = \frac{u_{is}^{(m)}}{u_{ic}^{(m)}}. \tag{38}$$

After the foregoing mathematical preparation, the true (real) values of the sought quantities are determined by the expression

$$\{Q_{ij}^{(m)}, u_i^{(m)}\} = \mathrm{Re}(\{\tilde{Q}_{ij}^{(m)}, \tilde{u}_i^{(m)}\mathrm{e}^{\mathrm{i}\omega t}\}), \tag{39}$$

according to which,

$$Q_{22}^{(m)} = |\tilde{Q}_{22}^{(m)}|\cos(\alpha_{22}^{(m)} + \omega t); \quad Q_{11}^{(m)} = |\tilde{Q}_{11}^{(m)}|\cos(\alpha_{11}^{(m)} + \omega t),$$

$$Q_{21}^{(m)} = -|\tilde{Q}_{21}^{(m)}|\sin(\alpha_{21}^{(m)} + \omega t); \quad Q_{12}^{(m)} = -|\tilde{Q}_{12}^{(m)}|\sin(\alpha_{12}^{(m)} + \omega t),$$

$$u_2^{(m)} = |\tilde{u}_2^{(m)}|\cos(\alpha_2^{(m)} + \omega t), \quad u_1^{(m)} = -|\tilde{u}_1^{(m)}|\sin(\alpha_1^{(m)} + \omega t). \tag{40}$$

It follows from expressions (38) and (40) that the functions $\alpha_{ij}^{(m)}(y_1, y_2^{(m)})$ and $\alpha_i^{(m)}(y_1, y_2^{(m)})$ are odd functions with respect to y_1. At the same time, by direct calculation, it is proven that

$$\alpha_{ij}^{(m)}(y_1, y_2^{(m)}) < 0, \quad \alpha_i^{(m)}(y_1, y_2^{(m)}) < 0 \quad \text{for } y_1 > 0,$$

$$\alpha_{ij}^{(m)}(y_1, y_2^{(m)}) > 0, \quad \alpha_i^{(m)}(y_1, y_2^{(m)}) > 0 \quad \text{for } y_1 < 0. \tag{41}$$

For fixed $y_2^{(m)}(= y_2^{(m)*})$ we can write

$$\alpha_{ij}^{(m)}(y_1, y_2^{(m)}) = \left.\frac{\partial\alpha_{ij}^{(m)}}{\partial y_1}\right|_{y_1=0} y_1 + \tilde{\alpha}_{ij}^{(m)}(y_1, y_2^{(m)}),$$

$$\alpha_i^{(m)}(y_1, y_2^{(m)}) = \left.\frac{\partial\alpha_i^{(m)}}{\partial y_1}\right|_{y_1=0} y_1 + \tilde{\alpha}_i^{(m)}(y_1, y_2^{(m)}). \tag{42}$$

It follows from (41) and from the foregoing discussions that

$$k_{ij}^{(m)} = \left.\frac{\partial \alpha_{ij}^{(m)}}{\partial y_1}\right|_{y_1=0} < 0, \quad k_i^{(m)} = \left.\frac{\partial \alpha_i^{(m)}}{\partial y_1}\right|_{y_1=0} < 0. \tag{43}$$

According to Eqs. (38), (41)–(43), the expressions given in (40) can be presented through multiplication of two terms, one of which is $e^{i(k_{ij}^{(m)}y_1+\omega t)}$ or $e^{i(k_i^{(m)}y_1+\omega t)}$. This is similar to the expressions corresponding to the wave propagation along the Oy_1 axis. It should be noted that this "wave propagation" is considered in a moving frame of reference. In this case, as it moves, propagating a wave with angular frequency ω and wave-number $k_{ij}^{(m)}$, the observation point oscillates with angular frequency ω in a moving frame of reference. However, in a fixed frame of reference, according to the foregoing discussions and expressions, it oscillates with angular frequency $\bar{\omega} = \omega - k_{ij}^{(m)}V$. This follows from the relationship

$$e^{i\omega t}e^{i\alpha_{ij}^{(m)}} = e^{i\omega t}e^{i(k_{ij}^{(m)}y_1+\tilde{\alpha}_{ij}^{(m)})} = e^{i\omega t}e^{i(k_{ij}^{(m)}(y'_1-Vt)+\tilde{\alpha}_{ij}^{(m)})} = e^{i(\omega-k_{ij}^{(m)}V)t}e^{i(k_{ij}^{(m)}y'_1+\tilde{\alpha}_{ij}^{(m)})} \Rightarrow \bar{\omega} = \omega - k_{ij}^{(m)}V.$$

Hence, according to the expressions and inequalities (41)–(43), for a fixed frame of reference, the oscillation frequency $\bar{\omega}$ of the observation point determined by coordinates $y_1 > 0$ ($y_1 < 0$) increases (decreases).

This statement is known physically as the Doppler effect. Consequently, the foregoing discussions and results agree with the well-known results of acoustic-physics and prove the validity of the mathematical modeling used for the problem considered.

With this we restrict here the discussions regarding the Doppler effect. More detailed analysis of this effect for the problem considered can be the subject of other separate investigations.

Consider the description of the algorithm regarding the calculation of the integrals in expression (35). Numerical investigations show that, in general, within fixed values of the problem parameters for each value of V the quantities $u_{iF}^{(m)}$, $Q_{ijF}^{(m)}$ have singular points with respect to $sH^{(2)}$. The following is a consideration of the determination of these singular points.

According to the procedure for determination of the unknowns $A_1^{(1)}(s), \ldots, A_4^{(1)}(s)$, $A_1^{(2)}(s), \ldots, A^{(2)}(s)$ (see, Refs. [22,25]), the aforementioned singular points coincide with the roots of the equation

$$\det \|\alpha_{nm}(V(sH^{(2)}))\| = 0, \quad n; m = 1, 2, \ldots, 6 \tag{44}$$

in $V(sH^{(2)})$, where $\alpha_{nm}(V(sH^{(2)}))$ are the coefficients of the unknowns in the algebraic equation system.

Consequently, the order of the singularity (denoted by r) of integrated values coincides with the order of the roots of Eq. (44). It is known that in the case where $0 \le r < 1$ the integrals in (35) can be calculated by the use of normal well-known algorithms. In the case where $r = 1$, the calculation of integrals (35) and (36) are performed according to Cauchy's principal value. But in the case where $r > 1$ the integrals do not have any meaning and the velocity corresponding to this case is called the "critical velocity" at which a resonance type of phenomenon takes place. In this case the values of all parameters of the problem are fixed (except the load-moving velocity V) and for the given value of $sH^{(2)}$, the velocity V is determined from Eq. (44) as the root of this equation. In this way, the dependence between V and $sH^{(2)}$ is obtained, and the critical velocity corresponds to the case where $dV/d(sH^{(2)}) = 0$.

One of the main questions of moving load problems for layered materials in a subsonic state (34) is the determination of this critical velocity (denoted by $V_{cr.}$), and the influence of the problem parameters (one of which is an oscillating frequency) of these moving load problems is the determination of the stress–strain state in the mechanical system wherein $V < V_{cr.}$. In this case, integrals (35) and (36) are calculated with the algorithm developed in Refs. [22,23].

We attempt at this point to analyze an equivalency for the definition of the critical velocity used in the present paper and in papers [10,11]. As has been noted in Section 1, in paper [10] the critical velocity is defined as the velocity for which the phase and group velocities are equal to each other. Note that the definition of the critical velocity used in the present paper coincides with that proposed in [10]. This conclusion can be proven as follows (below instead of the notation V and s we will use the notation c and k, respectively, where c is a phase velocity and k is a wave number):

$$\begin{aligned} dc/d(kH^{(2)}) &= d(\omega/k)/d(kH^{(2)}) = d(\omega H^{(2)}/(kH^{(2)}))/d(kH^{(2)}) \\ &= 1/(kH^{(2)})(d\omega/dk) - 1/(kH^{(2)})^2(\omega/k) = 0 \Rightarrow c_g = d\omega/dk = \omega/k = c_p. \end{aligned}$$

Consequently, as in the case where $\Omega = 0$, the physical meaning of the critical velocity is defined by the equation $dV/d(sH^{(2)}) = 0$ (because under $\Omega = 0$ Eq. (44) coincides with the dispersion equation of the system considered) which is equivalent to that proposed in paper [10]. At the same time, in paper [11] the critical velocity is defined as a minimal phase velocity after which wave propagation starts to occur in the system analyzed. The determination $dV/d(sH^{(2)}) = 0$ of the critical velocity under $\Omega = 0$ also confirms that proposed in Ref. [11], if the velocity satisfying the equation $dV/d(sH^{(2)}) = 0$ is a minimal velocity. It is obvious that sometimes the velocity satisfying this equation may be also a maximal phase velocity. Therefore, in the authors' opinion, the definition and the physical meaning of the critical velocity based on the equality of the phase and group velocities is a more generalized one than the one based on the minimal phase velocity and also confirms the critical velocity used in the present paper, as well as in papers [2–6].

As has been mentioned above, the foregoing discussions concerning the critical velocity are for the case where $\Omega = 0$. However, in the case where $\Omega > 0$, in papers [10,11] the critical velocity is determined from the set of equations consisting of the dispersion equation plus the equation $\omega = kV \pm \Omega$ (the equation of the kinematic invariant) as well as the equation $\mathrm{d}\omega/\mathrm{d}k = V$ (i.e. the condition for the equality of the group velocity of radiated waves and the load velocity), and in this way two critical velocities are determined for each fixed Ω. In the present paper these critical velocities are also determined from the equation $\mathrm{d}V/\mathrm{d}(sH^{(2)}) = 0$ one of which corresponds to the values $sH^{(2)} < 0$, but the other one to the values $sH^{(2)} > 0$. It should be noted that in the case where $\Omega > 0$ the curves $V = V(sH^{(2)})$ determined from Eq. (44) do not coincide with the dispersion curves. Nevertheless, the expression $\psi^{(m)}$ in Eq. (27) shows that the curves $V = V(sH^{(2)})$ plotted for the values $sH^{(2)} > 0 (sH^{(2)} < 0)$ can be considered as the "dispersion curves" for the phase velocity determined by the expression $V = (\omega - \Omega)/k (V = (\omega + \Omega)/k)$. It follows from this statement that the definition of the critical velocity used in the present paper for the case where $\Omega > 0$ is equivalent to that used in papers [10,11]. The numerical examples for the foregoing discussions will be considered in the following section.

Thus we come to consider the numerical results obtained within the framework of the solution method discussed above and related to the influence of the oscillation frequency of the moving load, of the initial strain in the layers of the slab, and their mechanical properties on the values of the critical velocity and on the values of the stresses acting on the interface plane between the layers.

5. Numerical results and discussion

Assume that $\lambda^{(1)}/\mu^{(1)} = \lambda^{(2)}/\mu^{(2)} = 1.5$; all numerical investigations for this case have been made. For an illustration of the trustworthiness of the algorithm and PC programs used, first, we consider the case where $\Omega = 0$ which was also considered in Refs. [3,7,8] under $\mu^{(2)}/\mu^{(1)} = 2.0$, $\lambda_1^{(2)} = \lambda_1^{(1)} = 1.0$, $\rho^{(1)}/\rho^{(2)} = 0.5$. The influence of $H^{(1)}/H^{(2)}$ on the values of the critical velocity $c_{cr.}(= V_{cr.}/c_2^{(2)})$ is investigated. According to the well-known mechanical considerations, for the sub-sonic velocities of the moving load the values of $c_{cr.}$ must approach the corresponding results obtained in Refs. [3,7,8] with $H^{(1)}/H^{(2)}$. These numerical results are given in Table 1, and agree with the foregoing considerations and prove the trustworthiness of the numerical algorithm and PC programs used.

Analyses of the multiple numerical results show that the critical velocity of the moving load occurs in cases where $\mu^{(2)}/\mu^{(1)} > 1$. At the same time, the existence of the critical velocity depends also on the values of $\rho^{(2)}/\rho^{(1)}$ and on the values of $H^{(2)}/H^{(1)}$. This statement is illustrated by the data given in Table 2. In this table the sign "-" means that in the corresponding case the critical velocity does not exist. Moreover, it follows from Table 2 that the values of $c_{cr.}(= V_{cr.}/c_2^{(2)})$ decrease with $\mu^{(2)}/\mu^{(1)}$.

The foregoing results agree with the corresponding ones attained in Refs. [2,3]. An explanation of why the critical velocity occurs when $\mu^{(2)}/\mu^{(1)} > 1$ can be made by analyzing the nature of the dispersion relations of the wave propagation in the system considered. Under $\mu^{(2)}/\mu^{(1)} > 1$ these relations, i.e. the relations between $c/c_2^{(2)}$ and $kH^{(2)}$ (k is a wave number) have a local minimum. But in the case where $\mu^{(2)}/\mu^{(1)} < 1$, these relations do not have a local minimum.

Consider the case where $\Omega \times c \neq 0$. It should be noted that in the case where $\Omega \times c = 0$, taking the symmetry of the $\det \|\alpha_{nm}(c(sH^{(2)}))\|$ with respect to $sH^{(2)} = 0.0$ into account for analysis of the dependence $c = c(sH^{(2)})$ is sufficient for the consideration of this dependence in the interval $0 \le sH^{(2)} \le +\infty$ only. But, in the case where $\Omega \times c \neq 0$, according to the term $\Omega c\bar{s}$ in the expression $\psi^{(m)}$ in (27), this symmetry is violated. Therefore, in the case where $\Omega \times c \neq 0$ the analyses of the dependence $c = c(sH^{(2)})$ must be done in the interval $-\infty \le sH^{(2)} \le +\infty$. For illustration of the foregoing statement we consider the graphs of the dependence given in Fig. 2. These graphs are constructed under $\mu^{(2)}/\mu^{(1)} = 5.0$, $\rho^{(2)}/\rho^{(1)} = 1/6$, $\lambda_1^{(2)} = \lambda_1^{(2)} = 1.0$ for various values of Ω. It follows from this figure that the graphs constructed in the case where $\Omega = 0$ are symmetric with respect to the straight line determined by the equation $sH^{(2)} = 0$. However, as has been noted above, this symmetry is violated for the graphs constructed under $\Omega > 0$. The analyses of the numerical results show that up to certain

Table 1
The values of $c_{cr.}(= V_{cr.}/c_2^{(2)})$ for various values of $H^{(1)}/H^{(2)}$ under $\mu^{(2)}/\mu^{(1)} = 2.0$, $\lambda_1^{(2)} = \lambda_1^{(1)} = 1.0$, $\rho^{(1)}/\rho^{(2)} = 0.5$.

$H^{(1)}/H^{(2)}$						
0.5	1.0	1.5	2.0	4.0	6.0	∞
0.9084	0.8812	0.8651	0.8556	0.8431	0.8415	0.8415 [7,8] 0.8370 [3]

Table 2
The influence of the parameters $\mu^{(2)}/\mu^{(1)}$, $\rho^{(1)}/\rho^{(2)}$, $H^{(1)}/H^{(2)}$ on the values of $c_{cr.}(=V_{cr.}/c_2^{(2)})$ under $\lambda_1^{(2)}=\lambda_1^{(1)}=1.0$.

$\mu^{(2)}/\mu^{(1)}$	$\rho^{(2)}/\rho^{(1)}$	$H^{(1)}/H^{(2)}$			
		0.5	2.0	5.0	7.0
2	0.5 $c_2^{(2)}=c_2^{(1)}$	0.9098	0.8556	0.8419	0.8414
	2.0 $c_2^{(2)}>c_2^{(1)}$	–	–	–	–
	0.25 $c_2^{(2)}<c_2^{(1)}$	0.9160	0.8906	0.8891	0.8891
3	1./3. $c_2^{(2)}=c_2^{(1)}$	0.8777	0.7953	0.7760	0.7753
	0.5 $c_2^{(2)}>c_2^{(1)}$	–	0.7663	0.7307	0.7264
	0.25 $c_2^{(2)}<c_2^{(1)}$	0.8818	0.8096	0.7970	0.7968
5	0.2 $c_2^{(2)}=c_2^{(1)}$	0.8266	0.7142	0.6871	0.6859
	0.25 $c_2^{(2)}>c_2^{(1)}$	0.8235	0.7049	0.6728	0.6707
	1./6. $c_2^{(2)}<c_2^{(1)}$	0.8287	0.7201	0.6965	0.6957

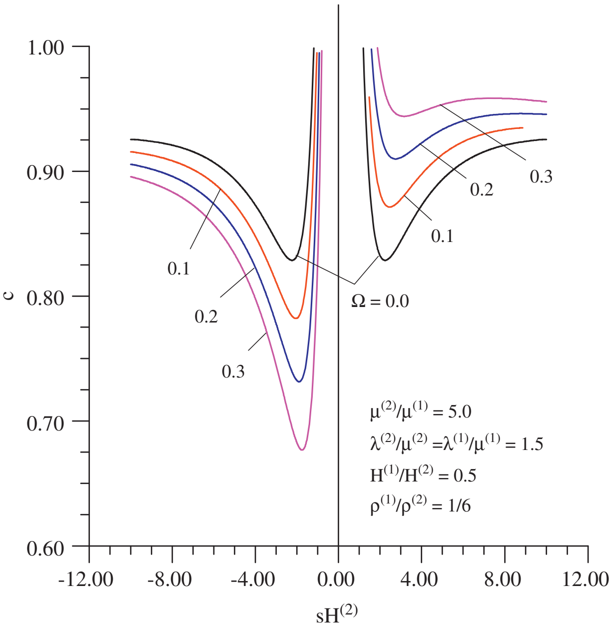

Fig. 2. The graphs of the dependencies between c and $sH^{(2)}$ for determination of the values of $c'_{cr.}$ and $c''_{cr.}$ for various values of Ω under $H^{(1)}/H^{(2)}=0.5$, $\lambda_1^{(2)}=\lambda_1^{(1)}=1.0$, $\mu^{(2)}/\mu^{(1)}=5.0$, $\rho^{(1)}/\rho^{(2)}=1./6$.

Ω (denoted by Ω') the values of the "minimum" critical velocity (denoted by $c'_{cr.}$) are determined by the left branches, i.e. by those branches attained in the region $sH^{(2)}<0$. The following values (by magnitude) of the critical velocity (denoted by $c''_{cr.}$) are determined by the right branches, i.e. by the branches obtained in the region $sH^{(2)}>0$. In this case the values of $c'_{cr.}(c''_{cr.})$ decrease (increase) with Ω. Consequently, up to a certain value of the oscillation frequency (i.e. up to $\Omega=\Omega'$) of the moving load, as a result of the oscillation of this load, the critical velocity decreases. However, in the case where $\Omega>\Omega'$ the left branches of the graphs do not determine the critical velocity; in other words, on these branches, the point for which the equation $dc/d(sH^{(2)})=0$ satisfies, as critical velocity does not occur, but the right branches of the graphs do determine the values of $c_{cr.}$ for the case where $\Omega>\Omega'$. This procedure continues up to a certain value of Ω (denoted by Ω'') after which the value of $c''_{cr.}$ goes outside of the framework of the subsonic moving regime. It should be noted that the numerical results shown in Fig. 2 in the qualitative sense agree with the corresponding results obtained in Refs. [10,11].

The influence of the problem parameters $\lambda_1^{(2)}$and $\lambda_1^{(1)}$ which characterize the magnitude of the initial strains in the layers of the slab is analyzed, and the influence of the parameter $H^{(1)}/H^{(2)}$ on the values of $c'_{cr.}$ attained for the various values of Ω.

Table 3
The influence of the parameters $\lambda_1^{(2)}$, $\lambda_1^{(1)}$ on the values of $c'_{cr.}$ for various values of $H^{(1)}/H^{(2)}$ in the case where $\Omega = 0.0$.

$\lambda_1^{(2)}/\lambda_1^{(1)}$	$H^{(1)}/H^{(2)}$		
	0.5	2.0	5.0
1.0/1.0	0.8266	0.7142	0.6871
1.02/1.0	0.8657	0.7646	0.7393
1.05/1.0	0.9206	0.8326	0.8088
1.07/1.0	0.9551	0.8740	0.8505
1.0/1.02	0.8263	0.7152	0.6898
1.0/1.05	0.8259	0.7037	0.6771
1.0/1.07	0.8393	0.7068	0.6846
1.0/1.10	0.8399	0.7113	0.6937
1.0/1.15	0.8411	0.7184	0.7042
1.0/1.20	0.8427	0.7253	0.7113

Table 4
The influence of the parameters $\lambda_1^{(2)}$, $\lambda_1^{(1)}$ on the values of $c'_{cr.}$ for various values of $H^{(1)}/H^{(2)}$ in the case where $\Omega = 0.1$.

$\lambda_1^{(2)}/\lambda_1^{(1)}$	$H^{(1)}/H^{(2)}$		
	0.5	2.0	5.0
1.0/1.0	0.7800	0.6270	0.5603
1.02/1.0	0.8204	0.6793	0.6121
1.05/1.0	0.8772	0.7501	0.6805
1.07/1.0	0.9130	0.7932	0.7213
1.0/1.02	0.7794	0.6278	0.5639
1.0/1.05	0.7786	0.6128	0.5190
1.0/1.07	0.7950	0.6158	0.5285
1.0/1.10	0.7953	0.6203	0.5420
1.0/1.15	0.7963	0.6276	0.5625
1.0/1.20	0.7976	0.6348	0.5804

Table 5
The influence of the parameters $\lambda_1^{(2)}$, $\lambda_1^{(1)}$ on the values of $c'_{cr.}$ for various values of $H^{(1)}/H^{(2)}$ in the case where $\Omega = 0.2$.

$\lambda_1^{(2)}/\lambda_1^{(1)}$	$H^{(1)}/H^{(2)}$		
	0.5	2.0	5.0
1.0/1.0	0.7293	0.5268	0.3729
1.02/1.0	0.7709	0.5800	0.4188
1.05/1.0	0.8294	0.6521	0.4796
1.07/1.0	0.8663	0.6960	0.5161
1.0/1.02	0.7285	0.5272	0.3763
1.0/1.05	0.7273	0.5095	0.3812
1.0/1.07	0.7353	0.5121	0.3165
1.0/1.10	0.7779	0.5161	0.3287
1.0/1.15	0.8444	0.5228	0.3479
1.0/1.20	0.9022	0.5295	0.3658

The corresponding results are given in Tables 3–5 for the case $\Omega = 0.0$, 0.1 and 0.2, respectively. In this case it is assumed that $\mu^{(2)}/\mu^{(1)} = 5.0$, $\rho^{(1)}/\rho^{(2)} = 1./6$. The corresponding conclusions following these results will be given in the following section.

The numerical results regarding the distribution of the stresses $Q_{22}^{(2)}$ and $Q_{22}^{(1)}$ on the planes of the mid-layers are examined with respect to $y_1/H^{(2)}$. However, in this examination the following statements must be taken into account.

According to the discussions made in the previous section and the expressions (37) and (38), in the case where $\Omega \times c = 0$ the relations $\alpha_{ij}^{(m)} = 0$, $\alpha_i^{(m)} = 0$ occur. Consequently, in the case where $\Omega \times c = 0$ the distribution of the stresses $Q_{22}^{(2)}$ and $Q_{22}^{(1)}$ become symmetric with respect to $y_1/H^{(2)} = 0$. Moreover, in the case where $\Omega \times c = 0$ the distribution of the

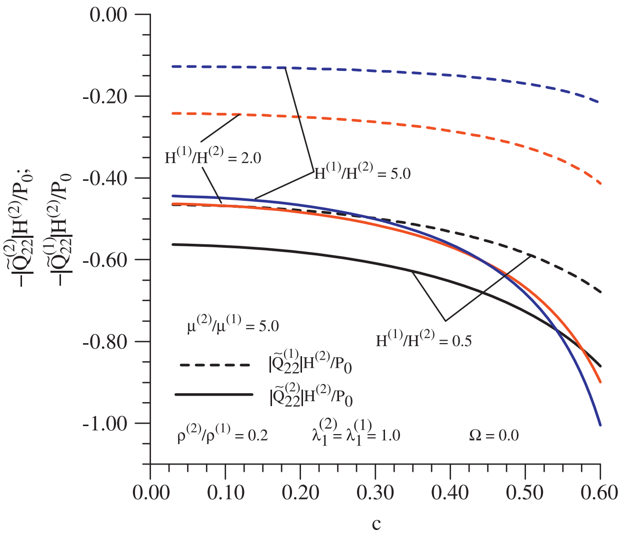

Fig. 3. The influence of the moving load velocity c on the values of $|\tilde{Q}_{22}^{(2)}|$ and $|\tilde{Q}_{22}^{(1)}|$ (at $y_1/H^{(2)}=0.0$) for various values of $H^{(1)}/H^{(2)}$ in the case where $\Omega=0.0$, $\lambda_1^{(2)}=\lambda_1^{(1)}=1.0$, $\rho^{(1)}/\rho^{(2)}=0.2$, $\mu^{(2)}/\mu^{(1)}=5.0$.

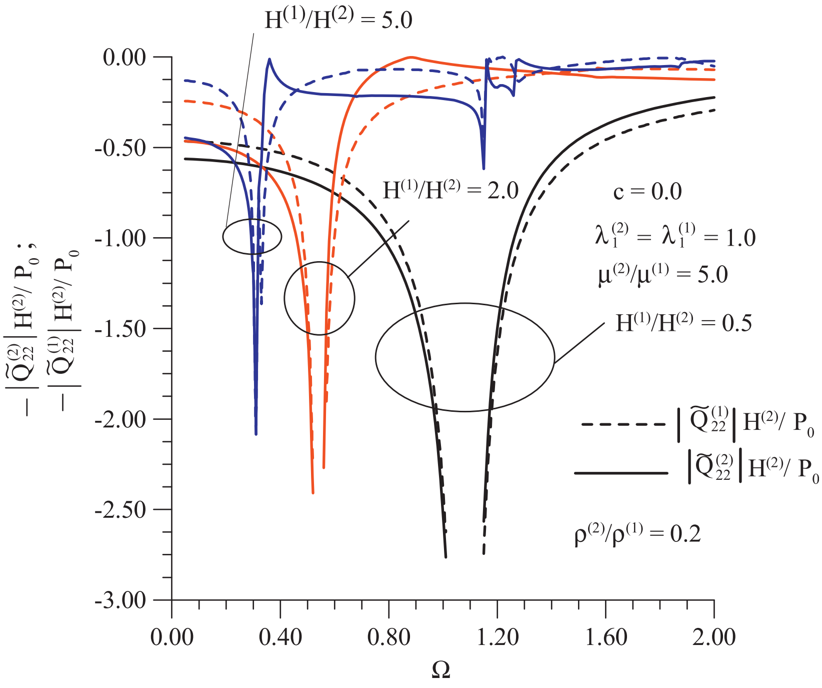

Fig. 4. The influence of the oscillating load frequency Ω on the values of $|\tilde{Q}_{22}^{(2)}|$ and $|\tilde{Q}_{22}^{(1)}|$ (at $y_1/H^{(2)}=0.0$) under various values of $H^{(1)}/H^{(2)}$ in the case where $c=0.0$, $\lambda_1^{(2)}=\lambda_1^{(1)}=1.0$, $\rho^{(1)}/\rho^{(2)}=0.2$, $\mu^{(2)}/\mu^{(1)}=5.0$.

$\tilde{Q}_{22}^{(2)}$ and $\tilde{Q}_{22}^{(1)}$, in the qualitative sense, also illustrate (simultaneously) the distribution of the $Q_{22}^{(2)}$ and $Q_{22}^{(1)}$, respectively. However, in the case where $\Omega\times c\neq 0$ the functions $\alpha_{ij}^{(m)}$ and $\alpha_i^{(m)}$ appear and these functions are discontinuous with respect to $y_1/H^{(2)}$. This discontinuity is explained by the wave reflection from the planes of the mid-layers. Moreover, in the case where $\Omega\times c\neq 0$, the distribution of the $Q_{22}^{(2)}$ and $Q_{22}^{(1)}$ becomes non-symmetric with respect to the straight line for which the equation is $y_1/H^{(2)}=0$. Nevertheless, the distribution of the $|\tilde{Q}_{22}^{(2)}|$ and $|\tilde{Q}_{22}^{(1)}|$ remain symmetric with respect to the straight line $y_1/H^{(2)}=0$ for the case $\Omega\times c\neq 0$.

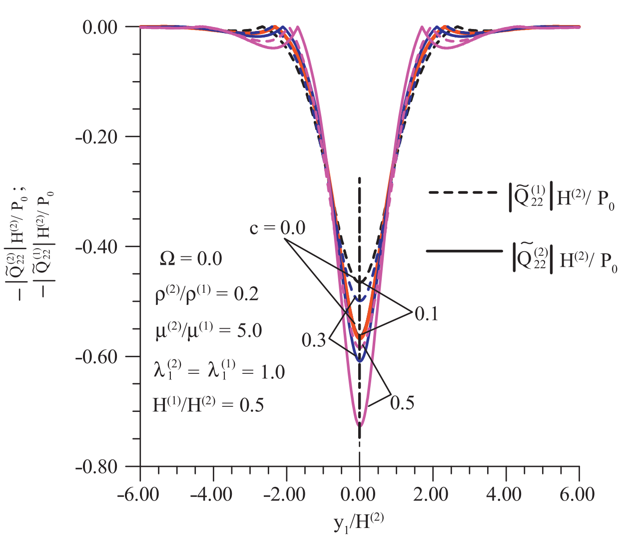

Fig. 5. The influence of the moving load velocity c on the distribution of $|\tilde{Q}_{22}^{(2)}|$ (at $y_2^{(2)} = -H_2^{(2)}$) and $|\tilde{Q}_{22}^{(1)}|$ (at $y_2^{(1)} = -H_2^{(1)}$) with respect to $y_1/H^{(2)}$, for $H^{(1)}/H^{(2)} = 0.5$, $\Omega = 0.0$, $\lambda_1^{(2)} = \lambda_1^{(1)} = 1.0$, $\rho^{(1)}/\rho^{(2)} = 0.2$, $\mu^{(2)}/\mu^{(1)} = 5.0$.

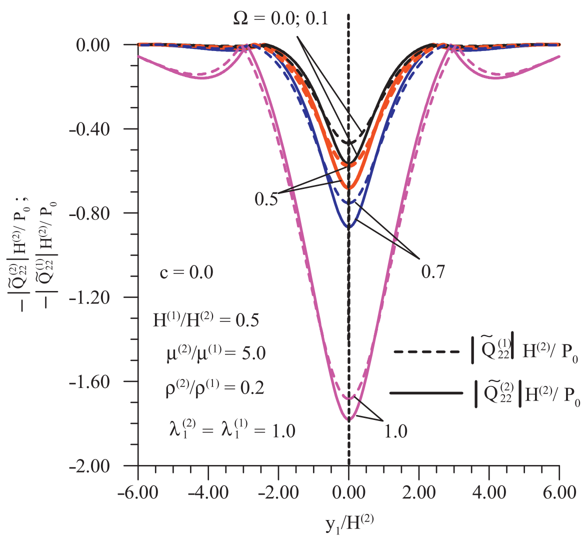

Fig. 6. The influence of the oscillating load frequency Ω on the distribution of $|\tilde{Q}_{22}^{(2)}|$ (at $y_2^{(2)} = -H_2^{(2)}$), $|\tilde{Q}_{22}^{(1)}|$ (at $y_2^{(1)} = -H_2^{(1)}$) with respect to $y_1/H^{(2)}$, for $H^{(1)}/H^{(2)} = 0.5$, $c = 0.0$, $\lambda_1^{(2)} = \lambda_1^{(1)} = 1.0$, $\rho^{(1)}/\rho^{(2)} = 0.2$, $\mu^{(2)}/\mu^{(1)} = 5.0$.

Thus, taking the foregoing discussions into account, consider the numerical results which are attained in the case where $\mu^{(2)}/\mu^{(1)} = 5.0$, $\rho^{(2)}/\rho^{(1)} = 0.2$. First, we examine the case where the initial strains in the layers of the slab are absent, i.e. in the case where $\lambda_1^{(1)} = \lambda_1^{(2)} = 1.0$, and investigate the dependence among the stresses $\tilde{Q}_{22}^{(2)}$(at $y_2^{(2)} = -H^{(2)}$, $y_1 = 0$), $\tilde{Q}_{22}^{(1)}(y_2^{(1)} = -H^{(1)}, y_1 = 0)$ and c under $\Omega = 0$, as well as among the foregoing stresses and Ω under $c = 0$. The graphs of these dependencies are given in Figs. 3 and 4 for various values of $H^{(1)}/H^{(2)}$.

It follows from the graphs given in Fig. 3 that the absolute values of the stresses $\tilde{Q}_{22}^{(2)}$ and $\tilde{Q}_{22}^{(1)}$ decrease with $H^{(1)}/H^{(2)}$. This result agrees well with the known mechanical considerations. At the same time, these graphs show that for all considered $H^{(1)}/H^{(2)}$ the absolute values of $\tilde{Q}_{22}^{(2)}$ and of $\tilde{Q}_{22}^{(1)}$ increase monotonically with c and $|\tilde{Q}_{22}^{(2)}|$; $|\tilde{Q}_{22}^{(1)}| \to \infty$ as $c \to c_{cr.}$.

Fig. 4 shows the resonance values of $\Omega(= \Omega_{res.})$ and these values decrease with $H^{(1)}/H^{(2)}$. This decrease can be explained with the decrease in the "stiffness" of the considered system with $H^{(1)}/H^{(2)}$.

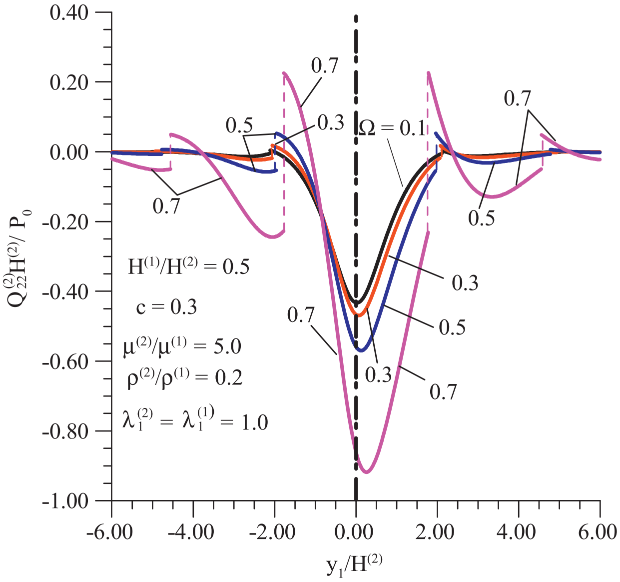

Fig. 7. The influence of the oscillating moving load frequency Ω on the distribution of the stress $\tilde{Q}_{22}^{(2)}$ (at $y_2^{(2)} = -H_2^{(2)}$, $\omega t = \pi/4$) with respect to $y_1/H^{(2)}$, for $H^{(1)}/H^{(2)} = 0.5$, $c = 0.3$, $\lambda_1^{(2)} = \lambda_1^{(1)} = 1.0$, $\rho^{(1)}/\rho^{(2)} = 0.2$, $\mu^{(2)}/\mu^{(1)} = 5.0$.

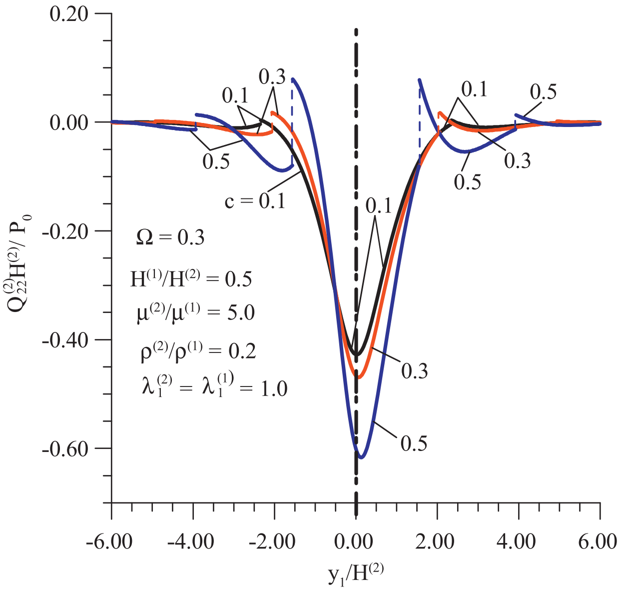

Fig. 8. The influence of the oscillating moving load velocity c on the distribution of the stress $\tilde{Q}_{22}^{(2)}$ (at $y_2^{(2)} = -H_2^{(2)}$, $\omega t = \pi/4$) with respect to $y_1/H^{(2)}$, for $H^{(1)}/H^{(2)} = 0.5$, $\Omega = 0.3$, $\lambda_1^{(2)} = \lambda_1^{(1)} = 1.0$, $\rho^{(1)}/\rho^{(2)} = 0.2$, $\mu^{(2)}/\mu^{(1)} = 5.0$.

Taking the foregoing results into account, we consider the distribution of the stress $\tilde{Q}_{22}^{(2)}$ and of $\tilde{Q}_{22}^{(1)}$ on the planes $y_2^{(2)} = -H^{(2)}$, $y_2^{(1)} = -H^{(1)}$ with respect to $y_1/H^{(2)}$ under $c < c_{cr.}$ and $\Omega < \Omega_{res.}$. The graphs of the distribution $-|\tilde{Q}_{22}^{(2)}|_{y_2^{(2)}=-H^{(2)}} H^{(2)}/P_0$, $-|\tilde{Q}_{22}^{(1)}|_{y_2^{(2)}=-H^{(2)}} H^{(2)}/P_0$ with respect to $y_1/H^{(2)}$ are given in Figs. 5 and 6 for the case where $\Omega = 0.0$ and $c = 0.0$, respectively. It follows from these graphs that the absolute dominant values of the stresses $|\tilde{Q}_{22}^{(2)}|$ and $|\tilde{Q}_{22}^{(1)}|$ arise in the region for which $|y_1/H^{(2)}| \leq 1.0$ under $\Omega = 0.0$ and in the region $|y_1/H^{(2)}| \leq 2.0$ under $c = 0.0$. At the same time, these values increase monotonically with c and Ω.

Consider the case where $\Omega \times c \neq 0$ and assume that $\omega t = \pi/4$. Fig. 7 shows the distribution of $Q_{22}^{(2)} H^{(2)}/P_0$ (at $y_2^{(2)} = -H^{(2)}$) with respect to $y_1/H^{(2)}$ under $c = 0.3$ for various Ω. The same graphs constructed under $\Omega = 0.3$ for various c

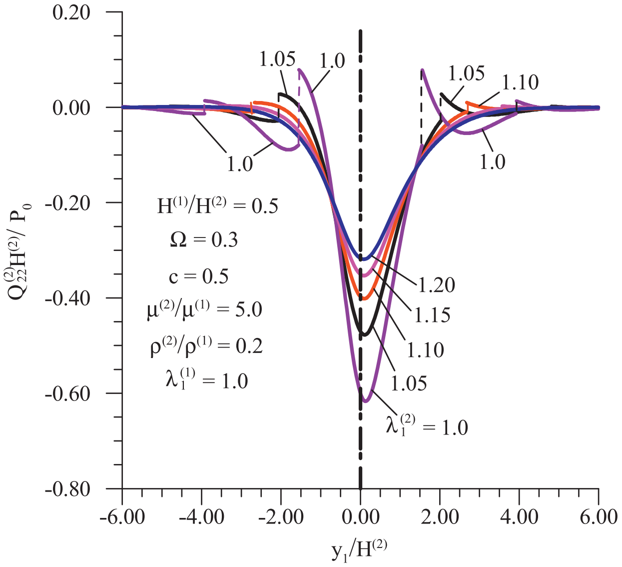

Fig. 9. The influence of the initial stretching of the upper layer of the slab, i.e. of the parameter $\lambda_1^{(2)}$ on the distribution of the stress $\tilde{Q}_{22}^{(2)}$ (at $y_2^{(2)} = -H_2^{(2)}$, $\omega t = \pi/4$) with respect to $y_1/H^{(2)}$, for $H^{(1)}/H^{(2)} = 0.5$, $c = 0.5$, $\Omega = 0.3$, $\lambda_1^{(1)} = 1.0$, $\rho^{(1)}/\rho^{(2)} = 0.2$, $\mu^{(2)}/\mu^{(1)} = 5.0$.

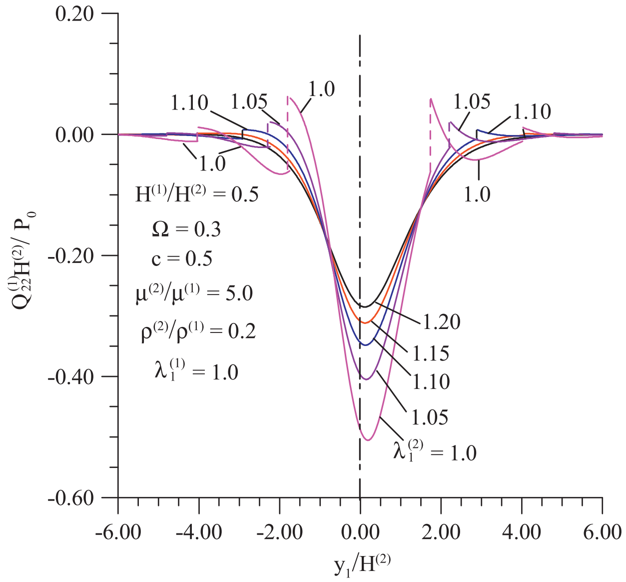

Fig. 10. The influence of the initial stretching of the lower layer of the slab, i.e. of the parameter $\lambda_1^{(1)}$ on the distribution of the stress $\tilde{Q}_{22}^{(2)}$ (at $y_2^{(2)} = -H_2^{(2)}$, $\omega t = \pi/4$) with respect to $y_1/H^{(2)}$, for $H^{(1)}/H^{(2)} = 0.5$, $c = 0.5$, $\Omega = 0.3$, $\lambda_1^{(2)} = 1.0$, $\rho^{(1)}/\rho^{(2)} = 0.2$, $\mu^{(2)}/\mu^{(1)} = 5.0$.

are given in Fig. 8. It follows clearly from these graphs that the considered distributions are non-symmetric with respect to $y_1 = 0$. The violation of the mentioned symmetries becomes more considerable with Ω (Fig. 7) and with c (Fig. 8). Moreover, the graphs show that the values of jumps at the discontinuity points increase with Ω and c. In these cases the discontinuity point approaches a point at which the external load acts with Ω as well as with c.

The influence of the initial strain of the upper layer of the slab on the distribution of $Q_{22}^{(2)}$ (at $y_2^{(2)} = -\lambda_2^{(2)}H^{(2)}/2$) and $Q_{22}^{(1)}$ (at $y_2^{(1)} = -H^{(1)}/2$) with respect to $y_1/H^{(2)}$ is investigated. It is assumed that $\lambda_1^{(1)} = 1.0$. The graphs of these distributions are given in Figs. 9 and 10 for the stresses $Q_{22}^{(2)}$ and $Q_{22}^{(1)}$, respectively. The graphs show that the aforementioned dominant values of the stresses decrease with $\lambda_1^{(2)}$, i.e. with the initial strains on the upper layer of the slab. Moreover, the results show that the values of a jump in the discontinuity point also decrease with $\lambda_1^{(2)}$, i.e. with the initial strains on the upper layer of the slab. Moreover, the results show that the values of the jump at the discontinuity point also decrease with $\lambda_1^{(2)}$ and this point moves away from the point at which the external force acts. Similar results are also obtained for other values of the problem parameters.

6. Conclusions

From the results analyzed above the following conclusions have been reached:

- The values of the critical velocity $c_{cr.}$ of the oscillating moving load decrease with the thickness of the lower layer of the slab.
- The critical velocity arises only in cases where the stiffness of the upper layer material is greater than that of the lower layer material. This conclusion agrees with the corresponding results attained in Refs. [2,3].
- As a result of the oscillation of the moving load, two types of critical velocity occur: one of them (denoted by $c'_{cr.}$) is lesser, but the other one (denoted by $c'_{cr.}$) is greater than the critical velocity attained for the case where the load moves without oscillation. This statement confirms in the qualitative sense the corresponding results obtained in Refs. [10,11].
- The initial stretching of the upper layer causes an increase in the values of $c'_{cr.}$ and $c''_{cr.}$; however, the influence of the initial stretching of the lower layer on the values of $c'_{cr.}$ and $c''_{cr.}$ is non-monotonic.
- The values of the critical velocity $c'_{cr.}(c''_{cr.})$ decrease (increase) with Ω, i.e. with the frequency of the oscillating moving load.
- The values of $c'_{cr.}$ and $c''_{cr.}$ decrease with the thickness of the lower layer of the slab and approach the corresponding values attained for the system consisting of a covering layer and half-plane.
- In the case where $c \neq 0$, $\Omega = 0$ (under $c < c_{cr.}$) and in the case where $\Omega \neq 0$, $c = 0$ (under $\Omega < \Omega_{res.}$), i.e. under action of the moving but non-oscillating load and under action of the oscillating but non-moving load, the dominant absolute values of the stresses $Q_{22}^{(2)}$ and $Q_{22}^{(1)}$ increase with c and Ω; at the same time, in these cases the distribution of the stresses $Q_{22}^{(2)}$, $Q_{22}^{(1)}$ has no discontinuity and is symmetric with respect to the point $y_1/H^{(2)} = 0$.
- In the case where $\Omega \times c \neq 0$ (under $c < c_{cr.}$, $\Omega < \Omega_{res.}$), i.e. in the case where the oscillating moving load acts on the bi-layered slab, the distribution of the stresses $Q_{22}^{(2)}$, $Q_{22}^{(1)}$ with respect to $y_1/H^{(2)}$ have discontinuity points, the jumps arising at these points increase with c and Ω.
- In the case where $\Omega \times c \neq 0$ the foregoing symmetry of the distribution of the stresses $Q_{22}^{(2)}$, $Q_{22}^{(1)}$ with respect to $y_1/H^{(2)}$ is violated. As a result of this violation the absolute maximum of the considered stresses occurs ahead of the point $y_1/H^{(2)} = 0$.
- The values of $\Omega_{res.}$ and $Q_{22}^{(2)}$, $Q_{22}^{(1)}$ decrease with $H^{(1)}/H^{(2)}$ and approach the corresponding ones obtained for the system consisting of a covering layer and half-plane.
- The initial stretching of the upper layer of the slab causes a decrease in the absolute dominant values of the stresses $Q_{22}^{(2)}$, $Q_{22}^{(1)}$.
- The values of the jumps at the discontinuity points decrease and these points go away from the point $y_1/H^{(2)} = 0$ (Fig. 1) at which the external load acts with the initial stretching of the upper layer of the slab.
- The results regarding the influence of the initial strains in the upper layer on the values of the critical velocity agree, in the qualitative sense, with the corresponding ones attained in [3,7–9]. The influence of the initial stretching of the lower layer of the slab on the distribution of the considered stresses is insignificant. Therefore, the numerical results regarding these cases are not considered here.

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